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Mathematics News Letter

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Dedicated to mathematics in general and to the following aims in particular: (1) a study of the common problems of secondary and collegiate mathematics teaching, (2) a true valuation of the disciplines of mathematics, (3) the publication of high class expository papers on mathematics, (4) the development of greater public interest in mathematics by the publication of authoritative papers treating its cultural, humanistic and historical phases.

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CONTENTS

MATHEMATICS A COMPLEX OF VARYING PROCESSES

It is well known that most efforts to define mathematics have been unsatisfactory to the majority of mathematicians. This is true even of definitions attempted by the mathematicians themselves. Is this fact partly due to the vast number of distinct phases of mathematics—some of them being so widely different in quality that different minds must necesarily view them with unequal degrees of interest? Peirce's definition that it is "the science of necessary conclusions manifestly excludes all other phases of the science except the logical one."

Other definitions have been offered in which the quantitative element of mathematics was made central. Yet, quantity and logic may be as wide apart as the earth's poles. On the other hand, an invisible and powerful bond may tie them together,—namely, a passion, belonging to many minds, which can be satified with nothing less than DEFINITE and CERTAIN knowledge. A knowledge which may be unattainable by the use of quantitative units alone, may yet be reached by a train of logical deductions fastened to these units.

So far as we know, no other science than mathematics embraces both the finite and the infinite aspect of things, that is, both entities which may be measured and those which may not. Then there is that aspect of mathematics which is called ORDER—a concept of essential importance to the science, yet one so widely different from the FINITE, the INFINITE, the QUANTITATIVE and the LOGICAL aspects that comparison of their qualities is out of the question.

Answering the question, what, then, is the underlying web which binds all these and many more varying aspects into the unity called mathematics, it can only be replied: The passion of some minds to know truth by whatever way it may be found, whether by measurement, by deduction, by ordered process applied to things finite or infinite, whether done singly or separately.—S. T. S.

DEVELOPING THE NEWS LETTER

Dear Professor Sanders:

Here, briefly, are some of the ideas which have come to mind concerning the second paragraph of your letter of March 8th—namely, how to attain some of the aims of the MATHEMATICS NEWS LETTER.

(1) Discussions concerning modernization of elementary mathematics, using only elementary methods. Example: use of vector methods in elementary plane and solid geometry and in analytic geometry. The French and Germans are doing this —most of our secondary teachers and many of our college teachers are not aware of how differently the Europeans present their material. I have not seen an American elementary text in which vectors, oriented angles, etc., are presented systematically—save Johnson's MODERN GEOMETRY.

(2) Clearing up of points which for various reasons are not properly or fully discussed in our usual college texts: examples, equivalence of equations, inequalities (both algebraic and trigonometric), consistent use of principal determination of radicals, especially square roots and the consequent limitations on the use of fractional exponents, principal determinations of inverse functions, elementary numerical analysis (i. e. using approximate values, obtained from tables, series, etc., how many figures are we warranted in keeping in our results).

(3) Examples of complete discussion of elementary problems.

(4) Expository papers concerning latest development in elementary fields, up through differential equations and elementary projective geometry.

(5) Discussion of teaching methods by both young and older teachers. One such discussion could be centered around a statement concerning specific ways in which to cause students to use scientific methods instead of non-scientifi- methods. E. g., many of my cadets insist upon putting $x=1$, $2, 3, \ldots$ in the equations of two curves, hoping by that method to tumble upon an integral value of x which makes the ordic nates of the two curves equal (and this is integral calculus!) How do various professors meet such difficulties? Viewed in this light do not many of our "popular" texts encourage, by their choice of problems, a wrong method of attack? Is it not

a fact that even if an instructor talks himself black and blue in the face that the average student will follow the book blindly and immediately forget the words of wisdom of the instructor? Another discussion might well be centered about what criterions different men use in the choice of a calculus or in fact any elementary text book. It would seem to me that if some such problems were formulated and if various schools were asked to have members of their department of mathematics write up a discussion of such things it might provoke quite a bit of friendly argument. Graduate students, high school teachers, and possibly even undergraduates might well enter such a discussion with profit.

(6) Simple discussions concerning concepts which are more or less vague to the young student. An account of the various ways in which the word "infinity" is used in elementary mathematics (real variable, complex variable, non-Euclidean geometry . . . .) might prove very interesting to all of us.

Of course you realize this is only a one-sided account of things that seem of interest to me. I hope you will find something of real value in these six topics after the waste material has been culled out.

I was interested in the reaction of Professor Seidlin to the MATHEMATICS NEWS LETTER. When you find time I should appreciate you reaction to this letter of mine.—W. E. B.

IN DEFENSE OF THE "INDIRECT" PROOF

By J. SEIDLIN
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Despite our era of rapidly changing high school curricula tenth grade geometry seems to have kept its content and method relatively intact. However, all present "signs and omens" point to a change. Teachers and lay lovers of geometry not actively engaged in proposing or promoting "greater freedom" or "reduction" or "fusion" are hopefully expectant though somewhat anxious. They realize that when the smoke of battle clears there will be "among the missing" many (of their) beloved theorems, schemes of organization, and methods of

proof. During periods of great agitation, preceding a change or changes, "attacks" on content and method outnumber "defenses". It is so in the case of the content and method of tenth grade geometry.

In particular, the indirect method of proof is being severely criticized and condemned.

I have culled the following sentiments from 88 papers "On the Direct Method" written by teachers of secondary school mathematics:

1. "The diagrams, as a rule, look silly, contradictory, and offensive to common sense and good eyesight."
2. ". . . and it doesn't really *prove*."
3. "What is the sense of proving a theorem by first disproving it?"
4. "A proof that contradicts the hypothesis is not good geometry."
5. "How can you talk about the analytic method in such glowing terms and so much as tolerate its antithesis (the indirect proof)?"
6. "It has never satisfied me. How can I expect it to appeal to my students!"
7. "Students never feel certain that they have exhausted all the possible situations."
8. " . . . and it certainly does not 'carry over'. It does not help to reason."
9. "It is the laughing stock of students."
10. "Since there are only three or four theorems proved by the Indirect Proof, it is hardly worth the fuss."

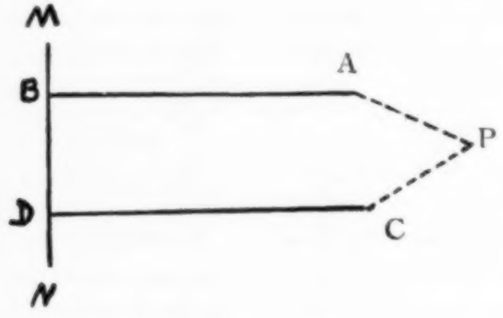
11-80. (And there were seventy more, perhaps somewhat less incisive, prosecuting opinions,—variations of the above.)

How shall we accept, or react to, the rather general contempt for the Indirect Proof? It would be neither wise nor expedient to browbeat so many otherwise bright students of geometry. What saves the day for us and for them is the fact that their reactions are not really to the Indirect Method, but to the incredible sort of thing that they were obliged to learn (or merely memorize) and, because of the textbooks they use, pass it on, in like distorted image, to their students.

The better to account for, and, in part, justify, the rather severe opinions previously quoted, let us examine two indirect proofs, given in exhibits M and N below, which are copied exactly from two popular textbooks.

Exhibit M.

"*Theorem.* Two lines perpendicular to the same line cannot meet however far produced.



(Fig. 1)

Given: AB $\perp$ MN at B
CD $\perp$ MN at D

To prove: AB cannot meet CD.

Proof: Suppose AB meets CD at P.

Then from P we would have PAB and PCD, both $\perp$ MN.

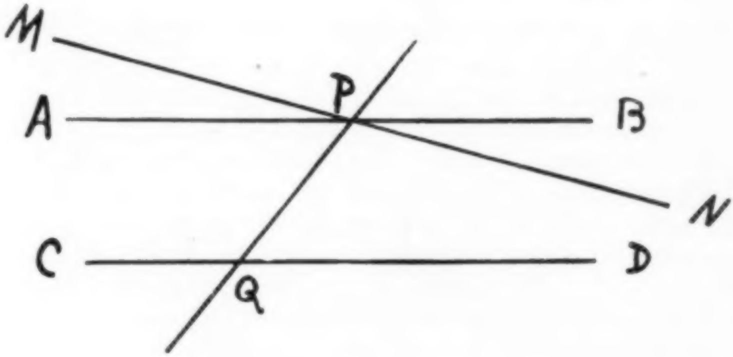
Since that is clearly impossible (only one perpendicular can be drawn to a given line from a given external point), it follows that AB cannot meet CD. Q. E. D."

Let us evaluate this proof from the pupil's point of view. To him the two lines AB and CD certainly *look* parallel, but AP and CP are obviously *not* prolongations of BA and DC. Furthermore, inasmuch as he refers to the diagram for guidance, it is prohibitively difficult for him to associate PAB and PCD with the two *straight* lines they are supposed to represent. Consequently, by the time the pupil reaches the statement "Since that is clearly impossible", he is quite ready to admit it—but not in the intended sense.

At the bottom of the page (footnote) we are told that the above illustrates the "reductio ad absurdum" type of proof. And it certainly does. Only it does not "lead to the ridiculous." It is ridiculous!

Exhibit N.

"*Theorem.* When two lines in the same plane are cut by a transversal, if the alternate-interior angles are equal, the two lines are parallel.



(Fig. 2)

Given the lines AB and CD cut by the transversal XY in the points P and Q respectively, so as to make the angles APQ and DQP equal.

To prove that AB $\parallel$ CD

Proof. Since we do not know that AB is $\parallel$ to CD, let us suppose MN drawn through P $\parallel$ CD.

We shall then prove that AB coincides with MN.

Now $\angle MPQ = \angle DQP$

(If two $\parallel$ lines are cut by a transversal, ...)

But $\angle APQ = \angle DQP$ Given

$\therefore \angle APQ = \angle MPQ$ Ax. 8

(Quantities that are equal to the same quantity . . .)

$\therefore$ AB and MN must coincide.

(Def. of equal angles)

But MN is || to CD Hyp. (?)

(for MN was drawn || to CD)

$\therefore$ AB, which coincides with MN, is || to CD.
Q. E. D."*

Normally we certainly would not expect any boy or girl to say (with the "aid" of the given diagram) "I don't know whether AB is or is not parallel to CD so I am going to suppose MN parallel to CD." Still less would we expect a normal pupil to continue: "I will then prove that AB (which looks as if it might be parallell to CD) coincides with MN (which even a blind man can see is not parallel to CD). Small wonder then that to the pupil the rest of the proof is artificial and unconvincing. For instance, it does not "follow clearly" that $\angle$MPQ = $\angle$DQP. As a matter of fact the pupil indicts the last statement on two counts: (1) He sees that $\angle$MPQ is greater than $\angle$APQ; (2) he knows that $\angle$APQ = $\angle$DQP.

In brief, almost every statement in the proof referred to the diagram is a tax on the pupil's better sense. The net result is a distrust of, as well as a positive dislike for, the indirect method. Need it seem so "incredible" even to the pupil? In particular, are misleading diagrams a necessary evil?

The following five *exhibits* are somewhat varying modes of treatment of the indirect method in which an attempt has been made to free these proofs of the "incredible" and the "misleading".

The reader's attention is directed particularly to the following: In no case is the first step of the proof an assumption that "the theorem is false".† The "other possibilities" are not given the preference of

*The theorem is given in the early part of the text. About 35 pages later we find the description of the proof: "THE INDIRECT METHOD OF PROOF. The method of proof that assumes the proposition false and then shows that this assumption is absurd is called the *indirect method* or the *reductio ad absurdum*. This method forms a kind of last resort in the proof of a proposition, after the syntehtic methods have failed."

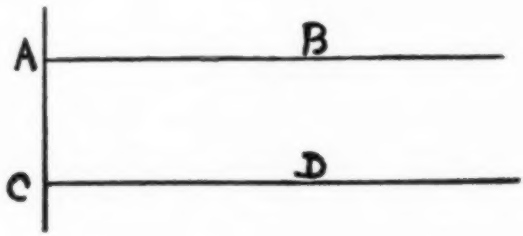
†In the opinion of the writer no such assumption can logically be made. A theorem consists of two parts,—the hypothesis and the conclusion. We know that the hypothesis is incontrovertible. The conclusion, as such, is merely a statement *to be proved*. In other words, it is a statement which we have no right to assume as either true or false. Consecquently, "assume the *theorem* false" is quite a meaningless injunction.

greater plausibility. That is, throughout there is rather a contentious attitude toward the other possibilities. Far from assuming them true, we undertake (from the start) to show that they are not true. Furthermore, ludicrous diagrams are avoided.

The "discussion" in Exhibit A while not an integral part of the proof is suggestive of a plausible introduction to, or as motivation for, the indirect method of proof.

Exhibit A.

**Theorem.* Two lines in the same plane perpendicular to the same line cannot meet however far they are produced.



(Fig. 3)

Given: $AB \perp AC$
$CD \perp AC$

To prove: AB cannot meet CD however far produced.

Discussion: On first thought it does not seem possible that even Demonstrative Geometry would attempt to prove that two lines *never meet*. Perhaps we feel like saying that AB and CD will not meet on this page or even at 100 yards from AC. But how about a mile from AC or 100 miles? What is there in the fact that AB and CD are perpendicular to AC that would prevent them from meeting at any point anywhere?

*Preliminary to the theorem we establish, or merely review, the fact that from a given point *one and only one* line can be drawn perpendicular to a given line.

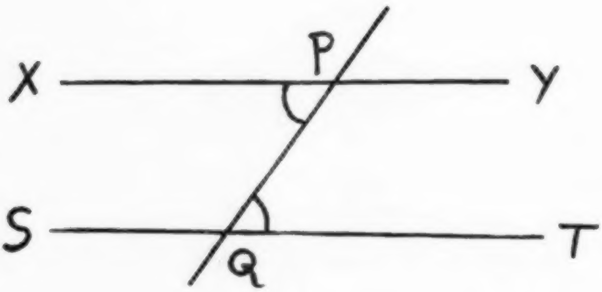
Proof: AB cannot meet CD however far produced. (For from a given point—no matter how far away—ONE AND ONLY ONE line can be perpendicular to AC). Q. E. D."

The "discussion" in Exhibit B is intended to show that the indirect method is intrinsically "analytic". The ever-growing number of teachers who espouse the analytic method may choose to incorporate the "discussion" as an integral part of the proof. Those who prefer the straight synthetic treatment will naturally omit the "discussion."

The mode of treatment here employed is perhaps the least desirable for the average pupil. For the latter, Exhibit C, gives the most practical and satisfying proof.

Exhibit B.

**Theorem.* When two lines in the same plane are cut by a transversal, if the alternate-interior angles are equal, the two lines are parallel.



(Fig. 4)

Given: XY and ST cut by PQ so that $\angle P = \angle Q$

(Where $\angle P$ is $\angle XPQ$; $\angle Q$ is $\angle PQT$)

To prove: XY cannot meet ST

*In particular it has been established that: (1) An exterior angle of a triangle is greater than either opposite interior angle; (2) Three non-concurrent intersecting lines form a triangle.

Discussion: We could show that XY cannot meet ST if we could show that XY, ST, and PQ do not form a triangle. We could show the latter if we could show that $\angle P$ is not greater than $\angle Q$ nor less than $\angle Q$. But we know that.

Proof: 1. $\angle P = \angle Q$ (Given)

2. $\therefore \angle P \ngtr \angle Q$; $\angle P \nless \angle Q$ (By 1.)

3. $\therefore \angle P$ cannot be the exterior angle of a triangle of which $\angle Q$ is the opposite interior angle (or vice versa) (By 2. and . . . .)

4. $\therefore$XY, ST, and PQ do not form a triangle. (By 3.)

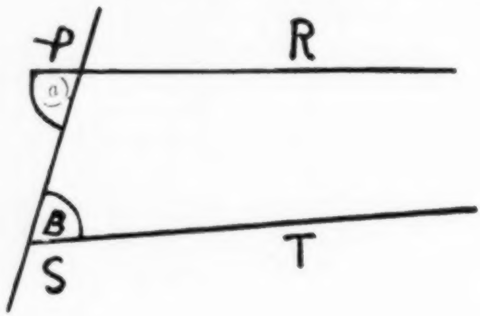
5. $\therefore$XY cannot meet ST (3 non-concurrent, etc.)
Q. E. D.

Thus far the impossible looking diagrams have been entirely avoided. In Exhibit C the "unlikely" is introduced in the least offending way. As a first shock absorber two diagrams are employed. The short line segments in the first diagram extended to meet at X in the second are inartistic but, as a complete picture, not glaringly improbable.

As in Exhibits A and B, the introductory, informal part of the proof is a justification of the indirect method as well as the preparation for it. Furthermore, in the first step, instead of confusing the pupil by saying "Let PR and ST intersect at some point", he is challenged to continue the investigation by "Were PR and ST to intersect at some point . . ."

Exhibit C.

Theorem: Two lines, in the same plane, cut by a transversal so that the alternate-interior angles are equal, are parallel.



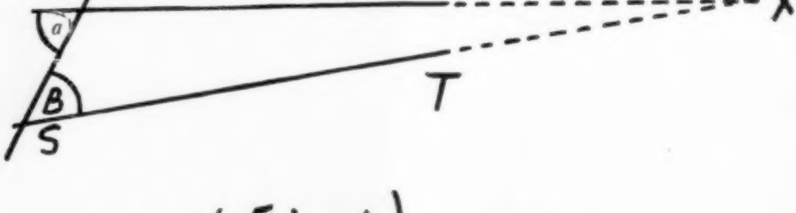
(Fig. 5)

Given: PR and ST cut by PS, so that $\angle a = \angle B$

To prove: PR || ST

Proof: In a given plane any pair of (distinct) lines, such as PR and ST, MUST be one of two kinds: (1) Either they meet at a point, or (2) they are parallel. We are going to establish (2) by showing that (1) cannot be true.

1. Were PR and ST to intersect at some point at the right of PS we should obtain the following figure:



(Fig. 6)

(3 non-concurrent intersecting lines form a triangle)

2. In the $\triangle$ PSX, $\angle a > \angle B$

(An ext. angle of a triangle, etc. . . .)

3. But $\angle a \ngtr \angle B$

(*given* $\angle a = \angle B$)

4. $\therefore$ PR cannot intersect ST at the right of PS. (Since with PS they (PR and ST) cannot form a triangle) (In like manner it may be proved that PR cannot intersect ST at the left of PS)

5. $\therefore$ PR $\parallel$ ST.

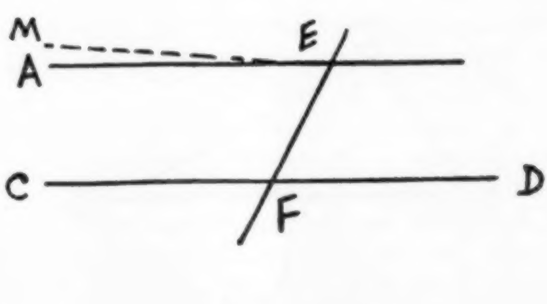
(A pair of lines in a plane MUST be either . . .)

Q. E. D.

Exhibit D is a modification of the most severely (and justly) criticized type of indirect proof. Let the reader review the proof reproduced on page 4, then read Exhibit D and note that the "method" is essentially the same in both. The latter (Exhibit D) is given as added evidence that most of the objectionable features of the proof on page 14 are NOT a necessary concomitant of the Indirect Method. On the other hand, the writer is convinced that the line of attack in Exhibit D is inferior to that of Exhibit C.

Exhibit D.

Theorem: When two lines in the same plane are cut by a transversal, if the alternate-interior angles are equal, the two lines are parallel.



(Fig. 7)

Given: $\angle AEF = \angle EFD$

To prove: AE || CD

Proof: We know that through E there exists one (and only one) line parallel to CD.

1. Let ME be that line (i. e. let ME || CD)
2. Then $\angle MEF = \angle EFD$ (alt.-int. $\angle s$ of || lines)
3. $\therefore \angle MEF = \angle AEF$ (Each is = to $\angle EFD$)
4. $\therefore$ME, if actually drawn parallel to CD, must coincide with AE (Def. of equal angles)
5. $\therefore$ AE || CD (AE coincides with ME)

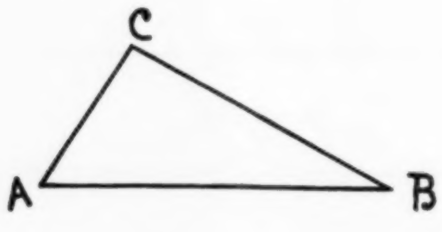
Q. E. D.

To the opponents of the Indirect Proof the least objectionable manifestation of it seems to be "proof by elimination". "Here," they say, "we do not have to 'swallow' offending diagrams." Their main objection seems to be to the fact that "the pupil is plunged too abruptly into '1. Suppose BC < CA'."

The "discussion" in Exhibit E eliminates the above objection and, also, enables us "to eliminate the two other possibilities" with greater confidence and directness.

Exhibit E.

Theorem: If two angles of a triangle are unequal, the sides opposite these angles are unequal and the side opposite the greater angle is the greater.



(Fig. 8)

Given: $\triangle ABC$ with $\angle A > \angle B$.

To prove: $BC > CA$.

Discussion: BC and CA, or any two lines, must be related to each other in one of three ways:

(1) $BC < CA$

(2) $BC = CA$

(3) $BC > CA$

No two of these can be true simultaneously; but one of the three MUST be true.

Proof: 1. $BC \not< CA$. For if BC were less than CA, $\angle A$ would be less than $\angle B$. (In a $\triangle$ the greater angle is opposite the the greater side), whereas we know that $\angle A > \angle B$ (given)]

2. $BC \neq CA$. [For if BC were equal to CA, $\angle A$ would equal $\angle B$. (In an isosc. $\triangle$ the base angles are equal), whereas we know that $\angle A > \angle B$ (given).]

3. $\therefore BC > CA$. [One of the three relationships—(1), (2), (3)—MUST HOLD.]

Q. E. D.

Undoubtedly there are students whose heaven-given I. Q. forever bars them from the kingdom of such poignant reasoning. These may "laugh at it" in sheer desperation. Others can do no less than admire it.

Would, or could, the pupil, unaided, discover the above proof, or any of the indirect proofs? Most likely not! But is not "discovering proofs", whatever the method, a mentally super-exacting recreation? To quote an authority on the teaching of mathematics:* "When the dominating aim is the discovery of proofs, the question of how to find them becomes of capital importance. There is however, no royal road to the discovery of mathematical results. Original

*J. W. A. Young: The Teaching of Mathematics (Reprinted 1925), page 78.

discoveries by the great men of this science no doubt require a special aptitude and training, and involve the element of good fortune as much as discoveries in other sciences. Discoveries in this sense cannot be made by the pupil."

Are the pupils ever certain that they have "eliminated" all but one of the totality of possibilities? It is probably true that some pupils remain unsatisfied. Just as it is true that to some people the statement "X is not in New York" more certainly denotes the absence of X from New York than the statement "X is in Philadelphia". Needless to say, to such people the indirect proof will ever remain unconvincing. But then I have known college students who accept certain axioms at considerably less than par value of "self-evident truth". And so we have genuine questions like the following: "Are the roots of $2x^3-4x^2+6x-10=0$ the same as the roots of $x^3-2x^2+3x-5=0$?" Or, "are the roots of $-x^5+x^3-2x-5=0$ the same as the roots of $x^5-x^3-2x+5=0$?" Yet teachers would not be likely to suggest that the axiom "If equals are multiplied by equals, the products are equal", needs "looking into" since it fails to satisfy some students.

The indirect method fails in cases of unlimited or unknown "possible conclusions." And, by the way, that is precisely why the indirect method is so often abused in the hands of the naive, the ignorant, or the unscrupulous. On the other hand it seems very desirable that indirect proof should be clearly understood in our high school geometry classes because this same method of proof is so frequently used in everyday life situations. The legal advice by which an accused person shows that he is innocent by proving an *alibi* is an excellent illustration of indirect reasoning.

Is the Indirect Proof "analytic"? A method is said to be analytic if it employs the "If I could show that, then I could prove this "couple". Is a method any less "analytic" in employing, with equal eflect, the "If I could show that that is not so, then I could prove that this is not so" couple? It seems to me that the Indirect Proof in "Exhibit B", page 8, is genuinely analytic.

It is perhpas unnecessary to review the content of this paper in order to show the collapse of the more pertinent objections to the Indirect Proof. It may be sufficient to point out afresh that the cause

of the objections is most likely due to first contacts with a misrepresentation of the Indirect Proof.

To eliminate "reasoning by elimination", would be to impoverish the content of (the logic of) geometry as well as to deprive it of a vital connecting link with the "outside world". Shall we condemn a method merely because it has been needlessly disfigured by textbook writers!

SUMMING SERIES WHOSE GENERAL TERMS ARE POLYNOMIALS

By WILSON L. MISER
Vanderbilt University

Several years ago when the author was instructing a class in college algebra, a freshman asked how to find the sum of such a series as

$$1^2+2^2+3^2+\ldots+n^2. \tag{1}$$

The topic before the class was mathematical induction. The sum (1) was given by the textbooks as

$$\tfrac{1}{6}n(n+1)(2n+1).$$

This sum was readily proved to be correct by mathematical induction. As to how such an expression could be written down by inspection or by any method at all was not mentioned in the textbook. The freshman insisted that there should be some method.

By the formula for the sum of n terms in an arithmetical progression, it was easily shown that

$$1+3+5+\ldots+(2n-1)=n^2. \tag{2}$$

But the sum of series (1) could not be derived by any formulas in the progressions.

Consider the sum of the first n odd integers and write the sum as a quadratic in n,

$$1+3+5+\ldots+(2n-1)=a+bn+cn^2, \tag{3}$$

where the coefficients a, b, c are to be determined.

Letting $n=1,\ 1=a+b+c,$

$n=2,\ 4=a+2b+4c,$

$n=3,\ 9=a+3b+9c.$

Solving these equations, the values are

$$a=0,\ b=0,\ c=1.$$

Hence the quadratic n^2 is found to be as given in (2).

It may be remarked that any arithmetical progression may be summed to n terms without finding the nth term by taking the sum in the quadratic form $a+bn+cn^2$ and by determining a, b, c from the equations

$$n=1,\ a+b+c=?,$$

$$n=2,\ a+2b+4c=?,$$

$$n=3,\ a+3b+9c=?.$$

Consider now the series (1) and assume that its sum is a cubic. So write

(4) $$1^2+2^2+3^2+\ .\ .\ .\ +n^2=a+bn+cn^2+dn^3.$$

Letting $n=0\quad 0=a+0+0+0,$

$n=1,\ 1=a+b+c+d,$

(5) $n=2,\quad 5=a+2b+4c+8d,$

$n=3,\ 14=a+3b+9c+27d.$

These equations give the following values.

$$a=0,\ b=\tfrac{1}{6},\ c=\tfrac{1}{2},\ d=\tfrac{1}{3}.$$

Substituting the values of a, b, c, d in (4) the cubic that results reduces to

$$\tfrac{1}{6}n(n+1)(2n+1).$$

If the sum of (4) were taken in the form of a fourth degree polynomial, the coefficient of n^4 could be easily found to be zero from the five equations in a, b, c, d, e corresponding to the equations (5).

In general, if the generating term of a series to be summed to n terms is a polynomial in n of degree k, the sum is a polynomial of degree $k+1$. The proof of this theorem. is easily made by the calculus of finite differences. In any particular series, the sum may be proved correct by mathematical induction whenever the coefficients a, b, c, . . . have been found. So the freshman's question is fully answered.

SOME OBSERVATIONS ON THE ANCHOR RING

By W. V. PARKER
Mississippi Woman's College
Hattiesburg, Mississippi

If the circle $x^2+(y-b)^2=a^2$, $b>a$, be revolved about the x-axis it will generate an anchor ring. The surface area of this anchor ring is found by adding the surface, S_1, generated by the part of the circle below the line $y=b$ to the surface, S_2, generated by the part of the circle above this line. If we denote these parts of the circle by C_1 and C_2 respectively we have the following equations for them

$$C_1 : y=b-\sqrt{a^2-x^2}\ ,$$

$$C_2 : y=b+\sqrt{a^2-x^2}\ .$$

By the usual process of calculating the area of a surface of revolution we get

$$S_1=2\Pi\int_{-a}^{a}(b-\sqrt{a^2-x^2})\frac{a}{\sqrt{a^2-x^2}}dx$$

$$=2\Pi ab\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}-2\Pi a\int_{-a}^{a}dx=2\Pi^2ab-4\Pi a^2.$$

We note here that the area S_1 is equal to the area of a circular cylinder whose radius is b and whose height is Πa (the length of C_1) minus the area of a sphere of radius a (the area of the surface gen-

erated by rotating C_1 about the line $y=b$). Similarly S_2 is the sum of these two areas. The area S is, therefore, equal to the area of a circular cylinder whose radius is *b* and whose height is equal to the circumference of the circle.

Suppose now that we have a curve through the points (c, b) and $(d,b), b>0$, which satisfies the conditions that in the interval $c<x<d$, y is a single-valued continuous function of x such that $0<y<b$. The equation of this curve in this interval may then be written in the form $y=b-f(x)$. Denote by C_1 the segment of the curve in this interval. If now we rotate the area inclosed by $x=c, x=d, y=0$ and C_1 about the $x-$axis, the area S_1 of the part of the surface generated by C_1 is given by

$$S_1=2\Pi\int_c^d [b-f(x)]\sqrt{1+f'^2(x)}\,dx$$

$$=2\Pi b\int_c^d \sqrt{1+f'^2(x)}\,dx-2\Pi\int_c^d f(x)\sqrt{1+f'^2(x)}\,dx.$$

But $\int_c^d \sqrt{1+f'^2(x)}\,dx=$ length of C_1, and

$2\Pi\int_c^d f(x)\sqrt{1+f'^2(x)}\,dx=$ area of surface generated by revolving C_1 about the line $y=b$.

We have, therefore, that S_1 is equal to the area of a circular cylinder whose radius is *b* and whose height is the length of C_1 minus the ara of the surface generated by revolving C_1 about the line $y=b$. If now C_2 be the curve obtained by reflecting C_1 in the line $y=b$ and S_2 the surface generated by rotating C_2 about the x-axis we have S_2 equal to the sum of the above areas.

We see then that if a closed curve C, symmetric about the line $y=b$ and lying entirely above the x-axis be revolved about the x-axis, the area S of the surface generated is equal to the area of a circular cylinder whose radius is *b* and whose height is the length of C.

This is true even though C be made up of broken lines. Suppose for example that C is a rectangle with ends at right angles to the line $y=b$. The surface is then a cylindrical shell. If the inner and outer radii are r and R respectively we have $S=2\Pi(R+r)\,h+2\Pi(R^2-r^2)$, where h is the height of the cylinder. In this case $b=\dfrac{R+r}{2}$ and we have by the above

$$S=2\Pi\left(\frac{R+r}{2}\right)(2h+2R-2r).$$

These two values of S are obviously identical.

Suppose we revolve the area inclosed by the lines $y=r$, $x=d$ and $y-r=\dfrac{R-r}{d-c}(x-c)$ about the x-axis. The area of the curved surface (the part not generated by $y=r$) generated is the area of the frustum of a cone plus the area of a circular ring of width $R-r$. This area is from geometry

$$S=\Pi s(R+r)+\Pi(R^2-r^2), \text{ where } s \text{ is the slant height.}$$

From the above the cylinder will be of radius r and height $s+R-r$ and the surface of revolution about the line $y=r$ will be the entire surface of a cone of radius $R-r$ and slant height s. According to the above we have

$$S=2\Pi r(s+R-r)+\Pi s(R-r)+\Pi(R-r)^2.$$

Again these two values of S are readily seen to be identical

ON INVERSE FUNCTIONS

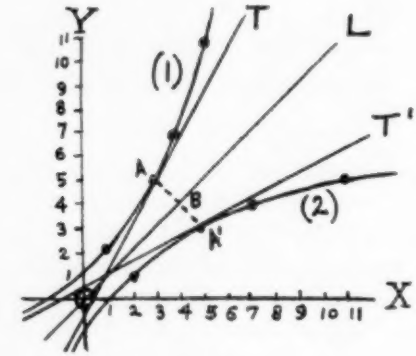
By W. PAUL WEBBER
Louisiana State University

The notion of inverse functions is of some importance even in elementary mathematics and its applications. Ordinarily the student is first introduced to the subject in trigonometry, and then through special methods that do not always arouse a suspicion that inverse functions may occur elsewhere in mathematical literature. It is proposed here to give a method of presentation that is general for all ordinary cases, and that is simple enough for high school and freshman college students to comprehend.

Consider any statistical table of number pairs as

1, 3, 4, 5,
2, 5, 7, 11.

If we take the numbers in the first row as abscissas, and those in the second row as ordinates, we can construct the graph, (1) in the figure.



Now if we exchange the roles of the numbers, and take these in the second row as abscissas and those in the first row as ordinates, we can construct the graph (2) in the figure. The latter graph we call the inverse of the first one.

Let A be any point on (1) and A′ the corresponding point on (2), and let OL bisect the quadrant XOY. Connect A and A′ by a straight line. Let B be the point of intersection of OL and AA′. It is easy to show that AA′ is bisected perpendicularly at B by OL. Since

A and A′ are any corresponding points of the two graphs the property just stated holds for all pairs of corresponding points. It is then obvious that if the figure is folded on OL as an axis, the two graphs can be made to coincide in every particular.

Suppose (1) has a tangent at A. Then (2) will have a corresponding tangent at A′. These tangents, also, will be symmetrical with respect to OL. It will now be easy to show that the angle Θ which AT makes with OX is equal to the angle Θ' which A′T′ makes with OY. It will follow that Θ and Θ' are such that $\tan \Theta' = 1/\tan \Theta$. This may be seen as follows: Let (x_1,y_1) ,(x_2,y_2) be two points on AT and let (X_1,Y_1) and (X_2,Y_2) be the corresponding points on A′T′. The slope of AT is

$$m = (y_2 - y_1)/(x_2 - x_1).$$

Similarly the slope of A′T′ is

$$m_1 = (Y_2 - Y_1)/(X_2 - X_1).$$

But when we recall that by the theory of our construction we have $Y_1 = x_1$ and $X_2 = y_2$, we see that $m_1 = 1/m_1$, as was to be shown. We have thus shown that the slope of the inverse curve is the reciprocal of the slope of the original curve.

It is obvious that the method is capable of generalization, and that if $y = f(x)$ is a function that has a derivative at any point A, the inverse function $Y = F(X)$ will ordinarily have a derivative at A′. We also shall have $dY/dX = 1/(dy/dx)$ for corresponding points. We now have the necessary theory to deal with all ordinary cases of inverse functions.

Consider the case of $y = \sin x$. The inverse may be written as $Y = \arcsin X$. As a particular value suppose $y = \sin 30°$. Then $Y = \arcsin .5$.

If it has been shown that when $y = \sin x$,$dy/dx = \cos x$, we can by the above theory pass to $Y = \arcsin X$ and $dY/dX = 1/\cos x$. But $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-y^2} = \sqrt{1-X_2}$. Hence, $dY/dX = 1/\sqrt{1-X^2}$. This is the usual formula of calculus.

Consider $y = x^2$. The inverse is, in our notation, $Y = \sqrt{X}$. We assume that we know that $dy/dx = 2x$. It follows that $dY/dX = 1/2x$. When we recall that $x = Y = \sqrt{X}$, we find immediately that $dY/dX =$

$1/2\sqrt{X}$. This is recognized as one of the usual forms of calculus. The same idea may be applied to $y=x^2$, and to higher powers.

Consider now the case of $y=e^x$, and its inverse $Y=\log_e X$. It is not difficult to convince a class that by successive substitutions a solution of the equation $e^x = N$ exists, where N is any positive real number. This will justify the use of a table of values of e^x. By selecting two or three pairs of values of x, each pair differing by not more that .01, it can be shown that the ratio $\Delta N/\Delta x$ is always nearly equal to e^x. A class will readily agree that this may justify, at least tentatively, that $d/dx(e^x)=e^x$. To obtain $d/dx(\log X)$ we have only to make use of the preceding method and arrive at the usual conclusion. In the case of the exponential this method may be found in a number of text books on calculus, but no particular general theory on which to base it is usually found.

It is seen that all the usual cases of inverse function are but particular cases of the generalization of our first example. The writer has found that by following the above general plan it is not difficult to "put over" the idea of inverse functions and their derivatives and that pupils attain a better grasp of the idea in general than by some of the particularized methods still to be found in some books. The correlation of the whole group of such functions under one principle at an early stage is helpful.

ON THE ROOTS OF A CUBIC AND THOSE OF ITS DERIVATIVE

By RAYMOND GARVER
University of California at Los Angeles

In the last two thirds of a century an interesting field of mathematics has been developed which is concerned with geometrical relations between the roots of a polynomial equation $f(x)=0$ and those of its derived equation $f'(x)=0$, the root $a+bi$ being plotted in the (x,y) plane as the point (a,b). In this country contributions to the development have been made by Bocher, Walsh, Echols, Curtiss, Van Vleck and others, and English, German, French, Scandinavian and Japanese mathematicians have shared in the work. However, in spite of the fact that many of the results are remarkably interesting they do not seem to be very widely known. The Bieberbach-Bauer

Algebra, for example, is the only book I am familiar with which gives much space to them, and I have not heard of their use in college or university courses in this country.

The best way to give an idea of the nature of these results is to state a few of them. Perhaps the best known, and certainly typical, are the Gauss-Lucas theorem and Jensen's theorem. The first of these is valid for polynomial equations $f(x)=0$ with complex coefficients, and states that any convex polygon which contains all the roots of $f(x)=0$ also contains all the roots of $f'(x)=0$. The word "contain" is here meant to allow points on the boundary, and the smallest convex polygon containing all the roots of $f(x)=0$ will, of course, have some of them on its boundary. In this case none of the roots of $f'(x)=0$ which are not at the same time roots of $f(x)=0$ will lie on the boundary however.

The Gauss-Lucas theorem can be inferred from a result stated by Gauss in 1816. Its first formal proof was given by Lucas, in a number of papers in the Comptes Rendus, about 1868-74. Lucas later collected his results in the Journal de l'Ecole Polytechnique, for the year 1879. Bocher gave a different proof in the Annals of Mathematics, vol. 7, 1892, and a number of proofs have appeared since.

Jensen's theorem is valid only for equations with real coefficients. The non-real roots of $f(x)=0$ then enter in conjugate pairs, and the line segment joining each such pair is used as the diameter of a circle. The circles thus obtained have been called the Jensen circles of the given equation. The theorem then states that no non-real root of the derived equation $f'(x)=0$ can lie outside all the Jensen circles of $f(x)=0$. This result was first stated by Jensen in the Acta Mathematica, vol. 36, 1912, p. 190. He did not give a proof, but it is hardly fair to imply, as some writers have done, that he had none. Proofs were published by Echols and Walsh at practically the same time; that by Echols may be found in vol. 27 of the American Mathematical Monthly.

This paper is concerned mainly with a more precise form of the Gauss-Lucas theorem which holds for cubic equations. The smallest convex polygon containing all the roots of $f(x)=0$ is then a triangle with the roots as vertices; we shall suppose that the roots do not lie on a straight line. The roots of the derivative quadratic can then be shown to be the foci of the ellipse which is inscribed in this triangle, and tangent to its sides at their mid-points. Lucas and Bocher both obtained this result, but not in a strictly elementary manner. Since the problem might well interest students in Calculus, or the Theory

of Equations, or even Analytic Geometry, it seems worth while to outline a simple proof which covers the case when the coefficients of the cubic are real. No such proof, has, I believe, been published.

First, we may, without loss of generality, take our cubic in the form $x^3+px+q=0$. In this problem this statement means that the configuration of five points connected with a non-reduced cubic, three points coming from the cubic and two from its derivative, is simply shifted as a whole, to the right or left, by the linear transformation which removes the second term of the cubic. Or, stated differently, the roots of the transform of the derivative are the same as the roots of the derivative of the transformed cubic. The student who has had the Calculus should see easily why this is so; a student in Analytic Geometry can obtain the result with a little computation. (The derivative of a polynomial might have to be defined for such a student.)

Now if a reduced cubic has only one real root its roots are of the form $2r$, $-r+si, -r-si$, where r and s are real. The cubic is then of the form $x^3+(s^2-3r^2)x-2r(r^2+s^2)=0$. The roots of the derivative equation are the square roots of the quantity $(3r^2-s^2)/3$, which we shall assume first to be positive, so that the roots lie on the real axis. It is now an easy matter to determine the equation of the ellipse having these two points as foci and having one vertex at $(-r, 0)$. This last condition will make the ellipse tangent to one side of the triangle at its mid-point. The equation is found to be

$$\frac{x^2}{r^2}+\frac{3y^2}{s^2}=1.$$

It then remains to be shown that this ellipse is tangent to the other side of the triangle at their respective mid-points. By symmetry, only one side need be considered. The side joining $(2r, 0)$ to $(-r, s)$ has its equation $sx+3ry=2rs$, and its mid-point $(r/2, s/2)$. It is easy to solve the equations of the line and ellipse simultaneously and show that they meet only at $(r/2, s/2)$. Or the Calculus may be used to obtain the equation of the tangent line to the ellipse at the point $(r/2, s/2)$. The proof is thus complete for the case when the roots of the derivative quadratic are real. The reader may show how it carries through when these roots are imaginary.

No result as simple as this holds for equations of higher degree. Walsh has, however, stated a theorem for certain quartics which the

reader can prove easily. If the roots of the quartic $f(x)=0$ are located at the vertices of a rectangle, then the circles whose diameters are the two longer sides of the rectangle intersect in roots of $f'(x)=0$. The remaining root of $f'(x)=0$ lies at the center of the rectangle.

Another interesting result of Walsh which can be proved easily for cubics is the following: If $f(x)=0$ is of degree n, with $n-2$ real roots and roots at i and $-i$, then all non-real roots of $f'(x)=0$ lie in or on the circle whose center is at $(0, 0)$ and whose radius is the square root of $(n-2)/n$. A cubic of this type must be $(x^2+1)(x-a)=x^3-ax^2+x-a=0$, where a is real. The derivative equation is $3x^2-2ax+1=0$, the product of whose roots is $1/3$. But if these roots are non-real, say $b+ci$ and $b-ci$, then the distance of either to $(0, 0)$ is $\sqrt{b^2+c^2}$, which is the same as the square root of the product of the two roots and is hence $\sqrt{1/3}$. Thus, in this case, the non-real roots of the derivative actually lie on (not in) the circle of Walsh's theorem.

INTEGRATION OF R(sinΘ, cosΘ)

By W. E. BYRNE
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Lexington, Va.

In connection with the article of Professor Smith in Volume 6, No. 3 of the *Mathematics News Letter* it might be interesting to consider some other transformations useful in the integration of rational functions of $\sin\theta$, $\cos\theta$, $R(\sin\theta, \cos\theta)$.

In what follows we shall use the notations $R(x,y)$, $R_1(x)$, $R_2(x)$. etc., to designate rational functions of the indicated arguments, and we shall write

$$f(\theta)\equiv R(\sin\theta, \cos\theta).$$

Theorem 1. The necessary and sufficient condition that $f(\theta)\equiv R_1(\tan\theta)$ is that $f(\theta)$ admit the period Π.

As an immediate corollary we see that

$$\int R(\sin\theta, \cos\theta)d\theta=\int R_1(u)\frac{du}{1+u^2}=\int R_2((u)du$$

where we have made the change of variable $u = \tan\theta$. Proof: That the condition is necessary is obvious. The condition is also sufficient. Replace $\sin\theta$ by $\tan\theta \cos\theta$ and $\cos^2\theta$ by $(1+\tan^2\theta)^{-1}$. After reduction $f(\theta)$ assumes the form

$$f(\theta) \equiv \frac{A+B\cos\theta}{C+D\cos\theta}$$

where A, B, C, D are polynomials in $\tan\theta$. By hypothesis

$$f(\theta) \equiv f(\theta+\Pi)$$

hence $$\frac{A-B\cos\theta}{C-D\cos\theta} \equiv \frac{A+B\cos\theta}{C+D\cos\theta} \equiv \frac{2A}{2C} \equiv \frac{A}{C} = R_1(\tan\theta)$$

provided C not $\equiv$ O. If $C \equiv 0$ and D not $\equiv 0$, we have

$$f(\theta) \equiv \frac{A+B\cos\theta}{D\cos\theta} \equiv \frac{A\cos\theta}{D(1+\tan^2\theta)^{-1}} + \frac{B}{D} \equiv A_1\cos\theta + B_1$$

where A_1, B_1 are rational functions of $\tan\theta$. But

$$f(\theta+\Pi) \equiv -A_1\cos\theta + B_1$$

so that $A_1 \equiv 0$ and

$$f(\theta) \equiv B_1$$

which is a rational function of $\tan\theta$. We see, then, that in both cases $f(\theta)$ reduces to a rational function of $\tan\theta$.

Theorem 2. The necessary and sufficient condition that

$$f(\theta) \equiv R_1(\cos\theta)\sin\theta$$

is that $f(\theta)$ be an odd function of θ.

As a corollary we have in this case

$$\int f(\theta)d\theta = -\int R_1(u)du$$

where $u = \cos\theta$.

Proof: The necessity is obvious. To prove the sufficiency we replace $\sin^2\theta$ by $1-\cos^2\theta$, obtaining

$$f(\theta) \equiv \frac{A+B\sin\theta}{C+D\sin\theta}$$

where A, B, C, D are polynomials in $\cos\theta$.

$$f(\theta) \equiv \frac{A+B\sin\theta}{C+D\sin\theta} \equiv -f(-\theta) \equiv -\frac{A-B\sin\theta}{C-D\sin\theta}$$

Hence, if C not $\equiv 0$

$$f(\theta) \equiv \frac{B\sin\theta}{C} \equiv R_1(\cos\theta)\sin\theta$$

Again, if $C \equiv 0$, D not $\equiv 0$

$$f(\theta) \equiv \frac{A\sin\theta}{D(1-\cos^2\theta} = \frac{A}{D_1}\sin\theta = R_1(\cos\theta)\sin\theta$$

Theorem 3. The necessary and sufficient condition that

$$f(\theta) \equiv R_1(\sin\theta)\cos\theta$$

is that

$$f(\Pi-\theta) \equiv -f(\theta)$$

Here a change of variable $u=\sin\theta$ is indicated. Proof: The necessity proof is trivial. The sufficiency follows easily as in the previous proofs. Replace $\cos^2\theta$ by $1-\sin^2\theta$ to obtain

$$f(\theta) \equiv \frac{A+B\ \cos\theta}{C+D\ \cos\theta}$$

where A, B, C, D are polynomials in $\sin\theta$.

$$-f(\Pi-\theta) \equiv -\frac{A-B\ \cos\theta}{C-D\ \cos\theta}$$

$$f(\theta) \equiv \frac{B\cos\theta}{C} \equiv R_1(\sin\theta)\cos\theta \text{ if } C \text{ not} \equiv 0.$$

If $C \equiv 0$ we have

$$f(\theta) \equiv \frac{A\cos\theta}{D(1-\sin^2\theta)} \equiv \frac{A}{D_1}\cos\theta \equiv R_1(\sin\theta)\cos\theta$$

and thus the proof is complete.

It happens frequently that the transformation

$$t = \tan\tfrac{1}{2}\theta$$

leads to a far more complicated integral than the transformations indicated above give.

It seems to me that the questions of this nature are passed over too lightly in most of the calculus texts on the market. In particular, attention is rarely directed to the fact that a definite branch of the inverse function must be decided upon if it is to be expected that consistent results are to follow.

ON THE DEFINITION OF THE SUM OF TWO VECTORS

By H. L. SMITH
Louisiana State University

The reader is familiar with the definition in ordinary vector analysis of the sum of two vectors. This definition may be stated vividly, but somewhat crudely, as follows: To add a vector B to a vector A, place the initial point of A upon the terminal point of A; then the vector whose initial point is the initial point of A and whose terminal point is the terminal point of B is the *sum* (vector) of A and B. This definition is available in ordinary vector analysis only for free vectors, that is, vectors which can be moved about in space without losing their identity.

The definition can be put into a form which involves no movement of a vector. This is as follows: If A, B, C are vectors having the same initial point, then C is the *sum* of A and B provided the mid-point of C is the same as the mid-point of the line segment whose end points are the terminal points of A and B.

For several months the writer has been engaged in developing a treatment of Riemann geometry in which all the fundamental concepts are more obviously geometric than in the usual treatment. Thus a vector was defined as an oriented geodesic arc, not as an ordered n-ple of numbers subject to a certain rule for transformation in case of a change of coordinate system. The law of cosines in trigonometry suggested an intrinsic definition of the cosine of the angle between two vectors, after which the scalar product of two vectors at the same point was defined as the product of the lengths of the vectors into the cosine of the angle between them.

The problem then arose of defining the sum of two vectors at a point in intrinsic fashion, that is, without using coordinates. Neither of the definitions given above could be used; the first because it was completely meaningless; the second because, on account of the curvature of the Riemann space, it did not correspond to the thing of which a definition was being attempted. A modification of the second definition given above at once suggests itself. It may be stated as follows: If A, B, C are vectors at the same point, then C is the sum of A and B provided that if d(t) is the distance between the mid-point of tC and the mid-point of the geodesic arc determined by the terminal points of tA, tB, and if l(t) is the sum of the lengths of tA and tB, then the limit of d(t)/l(t) is zero as t approaches zero. The writer has not investigated the suitability of this as a definition, but it seems reasonable to think that it would prove satisfactory. The reason for this belief is the fact that many infinitesimal figures in Riemann geometry behave as do finite figures in Euclidean geometry.

The following definition was finally adopted. If A, B, C are vectors at the same point, then C is the sum of A and B provided

$$AV = BV + CV$$

for *every* vector V at that point. Here AV represents the scalar product of vectors A and V, and BV, CV have similar meanings. On the basis of this simple definition a complete theory of linear independence and dependence was built.

PROBLEM DEPARTMENT

Edited by
T. A. BICKERSTAFF
University, Miss.

This department aims to provide problems of varying degrees of difficulty which will interest anyone who is engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and solve problems here proposed.

Problems and solutions will be credited to their authors.

Send all communications about problems to T. A. Bickerstaff, University, Miss.

A partial solution of problem No. 15 has been received from Mary Hamilton, Agnes Scott College, Decatur, Ga.

Solutions are desired for problems 2, 6, and 7.

Solutions

No. 17. Proposed by E. C. Kennedy, College of Mines and Metallurgy, El Paso, Texas:

Given $y = \sqrt{x - \sqrt{x - \sqrt{x -}}}$. ad inf.

Find y' .

Solved by the proposer:

Squaring,

$$y^2 = x - \sqrt{x - \sqrt{x - \sqrt{x -}}} \ldots .$$

$$= x - y$$

Deriving, we have

$$y' = \frac{1}{1+2y}$$

Problems for Solution

No. 21. Proposed by Everett E. Cook, University, Miss.:

A man died on a birthday before 1930 and his age then was $^1/_{29}$ of the year of his birth. How old was he on his birthday in 1900?

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